Outline: Curvature of Surfaces

1. **The Second Fundamental Form and Principle Curvatures**

   Let S be an oriented surface, and let p be a point on S. The **second fundamental form** of S at p is a certain quadratic form $I\!I$ that can be defined on the tangent space $\vec{T}_p(S)$. The eigenvalues κ_1, κ_2 of the second fundamental form are called the **principle curvatures** of S at p.

2. **Relation to the Hessian**

   Suppose that S is the graph of a function, i.e. a surface of the form $z = f(x, y)$ for some function $f\colon \mathbb{R}^2 \to \mathbb{R}$, with normal vectors pointing upwards. Let (x_0, y_0, z_0) be a point on S at which the tangent plane is horizontal. In this case, the second fundamental form of S at (x_0, y_0, z_0) is just the Hessian $Hf(x_0, y_0)$. In particular, the principle curvatures κ_1 and κ_2 are the eigenvalues of the Hessian.

   More generally, if S is any oriented surface and p is any point on S, we can rotate S until the tangent plane is horizontal. In this case, the portion of S near p looks like the graph of a function, and the second fundamental form of S at p is the Hessian of this function at p. Thus, the second fundamental form can be thought of as a "rotated" version of the Hessian.

3. **Gaussian and Mean Curvature**

   Let S be an oriented surface, and let p be a point on S. The **Gaussian curvature** and **mean curvature** of S at p are defined by the formulas

   $$K = \kappa_1 \kappa_2 \qquad \text{and} \qquad H = \frac{\kappa_1 + \kappa_2}{2},$$

   where κ_1 and κ_2 are the principle curvatures of S at p.

   The Gaussian curvature is also defined by the formula

   $$\vec{N}_u \times \vec{N}_v = K\, \vec{X}_u \times \vec{X}_v$$

   for any parametrization $\vec{X}(u, v)$.

   Geometrically, K is positive if S is like the surface of a sphere near p, and K is negative if S is more like the surface of a hyperbolic paraboloid near p.

4. **Principle Directions**

   The **principle directions** of a surface S at a point p are the directions of the eigenspaces of the second fundamental form. Thus the principle directions are a perpendicular pair of tangent directions to the surface at each point. For example, on a surface of revolution, the principle directions are the directions of the coordinate lines for the standard parametrization.